Problem 5086. Find the value of the sum

$$\frac{2}{3}+\frac{8}{9}+\cdots+\frac{2N^2}{3^N}$$

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The required sum can be written as $S_N = \dfrac{2}{3^N} \cdot x_N$, where x_n denote the sequence

$$x_n = 1^2 \cdot 3^{n-1} + 2^2 \cdot 3^{n-2} + 3^2 \cdot 3^{n-3} + \cdots + n^2 \cdot 3^0$$

Since

$$x_{n+1} = 1^2 \cdot 3^n + 2^2 \cdot 3^{n-1} + 3^2 \cdot 3^{n-2} + \cdots + n^2 \cdot 3^1 + (n+1)^2 \cdot 3^0$$

such a sequence satisfyies the linear recurrence

$$x_{n+1} - 3x_n = (n+1)^2 \tag{*}$$

Solving the characteristic equation $\lambda - 3 = 0$, we get the homogeneous solutions $x_n = A \cdot 3^n$, where A is a real parameter. To determine a particular solution, we look for a solution of form $x_n^{(p)} = Bn^2 + Cn + D$. Substituing this into the difference equation, we have

$$B(n+1)^2 + C(n+1) + D - 3\left[Bn^2 + Cn + D\right] = (n+1)^2 \qquad \Longleftrightarrow$$

$$-2Bn^2 + 2(B-C)n + B + C - 2D = n^2 + 2n + 1$$

Comparing the coefficients of n and the constant terms on the two sides of this equation, we obtain

$$B = -\frac{1}{2} \qquad , \qquad C = -\frac{3}{2} \qquad , \qquad D = -\frac{3}{2}$$

and thus

$$x_n^{(p)} = -\frac{1}{2}n^2 - \frac{3}{2}n - \frac{3}{2}$$

The general solution of (*) is is simply the sum of the homogeneous and particular solutions, i.e.

$$x_n = A \cdot 3^n - \frac{1}{2}n^2 - \frac{3}{2}n - \frac{3}{2}$$

From the boundary condition $x_1 = 1$, the constant A is determined as $\frac{3}{2}$.

Finally, the desired sum is

$$S_N = \frac{3^{N+1} - N^2 - 3N - 3}{3^N}$$

and we are done. □